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BOUNDS FOR EIGENVALUES OF HERMITIAN TRENCH MATRICES.(U)

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T. N. E. Greville

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BOUNDS FOR EIGENVALUES OF HERMITIAN TRENCH MATRICES

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T. N. E. Greville

Technical Summary Report #1977
July 1979

ABSTRACT

A banded matrix $H = (h_{ij})_{i,j=0}^{N}$ is one such that $h_{ij} = 0$ for $j - i > r$ and for $i - j > s$, where r and s are nonnegative integers. In [5] W. F. Trench and I called it strictly banded if, in addition, $r + s \le N$. We also showed that a necessary condition for a strictly banded matrix to have a Toeplitz inverse is that it have a certain special structure fully characterized by two polynomials, $A(x)$ of degree r and $B(x)$ of degree s. I call a matrix having this special structure a Trench matrix. It was also shown in [5] that a Trench matrix is nonsingular if and only if $A(x)$ and $B(x)$ have no common zero, and that a strictly banded matrix has a Toeplitz inverse if and only if it is a nonsingular Trench matrix. In this paper there are established bounds for eigenvalues of Hermitian Trench matrices that depend only on the polynomials $A(x)$ and $B(x)$ and not on the order of the matrix.

AMS(MOS) Subject Classification: 15A09, 15A57.

Key Words: Toeplitz matrix, Band matrix, Hermitian matrix, Matrix eigenvalue, Eigenvalue bound.

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SIGNIFICANCE AND EXPLANATION

A band matrix is one whose nonzero elements are confined to a diagonal band containing the principal diagonal. It is called strictly banded if the band width does not exceed the order of the matrix. A Toeplitz matrix is one in which all the diagonal elements are equal, and all the elements along each diagonal line parallel to the main diagonal are equal. In TSR #1879 W. F. Trench and I showed that a strictly banded matrix having an inverse that is a Toeplitz matrix must have a certain special structure fully characterized by two polynomials $A(x)$ and $B(x)$. I call a strictly banded matrix having this special structure a Trench matrix. If $A(x)$ and $B(x)$ have no common zero, the Trench matrix is nonsingular (and has a Toeplitz inverse). If they have a common zero, it is singular.

For some purposes it is useful to know bounds for the eigenvalues of a Trench matrix H. For example, in moving-weighted-average smoothing (see TSR #1786), the smoothing matrix G is of the form $G = I - kH$, where H is a singular Hermitian Trench matrix and k is a positive constant. G is called "stable" if the limit of G^n as $n \to \infty$ exists. Bounds for the eigenvalues of H can be used to determine whether G is stable.

In this paper we derive bounds for the eigenvalues of a Hermitian Trench matrix (whether singular or not) that depend only on $A(x)$ and $B(x)$ and not on the order of the matrix.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

BOUNDS FOR EIGENVALUES OF HERMITIAN TRENCH MATRICES

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1. Introduction.

In [5] W. F. Trench and I studied the conditions under which a band matrix has a Toeplitz inverse. More specifically, let $H = (h_{ij})_{i,j=0}^{N}$ be a real or complex matrix, where

$$h_{ij} = 0 \quad \text{for} \quad j - i > r \quad \text{or} \quad i - j > s ,$$

with

$$r \geq 0 , \qquad s \geq 0 .$$

Such a matrix we called a band matrix. We called it strictly banded if, in addition,

$$r + s \leq N .$$

Let

$$H_i(x) = \sum_{j=0}^{N} h_{ij} x^j$$

be the generating function of the elements of the ith row of H. In this paper I define a Trench matrix as a strictly banded matrix such that

$$H_i(x) = \begin{cases} x^i A(x) \sum_{\mu=0}^{i} b_\mu x^{-\mu} & (0 \leq i < s) \\ x^i A(x) B(1/x) & (s \leq i \leq N - r) \\ x^i B(1/x) \sum_{\nu=0}^{N-i} a_\nu x^\nu & (N - r < i \leq N) \end{cases} \tag{1.1}$$

where

$$A(x) = \sum_{\nu=0}^{r} a_\nu x^\nu , \qquad B(x) = \sum_{\mu=0}^{s} b_\mu x^\mu$$

are polynomials with real or complex coefficients (according as H is real or complex) and $a_0 b_0 \neq 0$.

Though the form (1.1) in its full generality previously appeared in a joint paper [5], it was first suggested by Trench, and, for a particular case,

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had been published by him in 1967 [8]. It is therefore appropriately called by his name.

By Lemma 3 of [5] a Trench matrix is nonsingular if and only if $A(x)$ and $x^s B(1/x)$ have no common zero. (Both real and complex zeros must be taken into account even if H is real.) In fact, it is shown in [5] that a strictly banded matrix has a Toeplitz inverse if and only if it is a nonsingular Trench matrix.

It is also shown in [5] that a Trench matrix is persymmetric: that is, symmetric about its secondary diagonal, and is also quasi-Toeplitz. The latter term implies that it has the Toeplitz property

$$h_{i+1,j+1} = h_{ij}$$

so long as neither of these elements is in the s by r submatrix in the upper left corner or the r by s submatrix in the lower right corner.

It is the purpose of this paper to establish certain bounds for the eigenvalues of Hermitian Trench matrices. More specifically, let the polynimials $A(x)$ and $B(x)$ be given, and consider the corresponding family of Trench matrices H_N given by (1.1) of all orders from $r+s+1$ to ∞. We wish to establish bounds depending on $A(x)$ and $B(x)$, but independent of N, for the eigenvalues of H_N. As there is an extensive literature on bounds for eigenvalues of Toeplitz matrices (see, e.g., [2], [9]), it is tempting to think that in the nonsingular case one could deduce bounds for the Trench matrices from what is known about their Toeplitz inverses. However, it turns out that this would impose severe restrictions on the choice of the polynomials $A(x)$ and $B(x)$.

Consider the family of Toeplitz matrices T_N characterized by the doubly infinite sequence $\{t_\nu\}_{\nu=-\infty}^{\infty}$ so that $T_N = (t_{ij})_{i,j=0}^{N}$, where $t_{ij} = t_{j-i}$, and note that, while the Trench matrices are banded, their Toeplitz inverses are not, so that the entire sequence $\{t_\nu\}$ is involved. The available theorems regarding bounds for eigenvalues of such families of Toeplitz matrices require that the Laurent series

$$\sum_{\nu=-\infty}^{\infty} t_\nu x^\nu \tag{1.2}$$

converge in some fashion in an appropriate region of the complex plane. The convergence may be weak (see, e.g., [9]), but we wish to extend our consideration to cases in which the series (1.2) does not exist or its convergence fails entirely.

Now, for the family of Toeplitz matrices whose inverses belong to the given family of Trench matrices, it was shown in [4] that (1.2) converges in some part of the plane if and only if all zeros of $x^s B(1/x)$ are smaller in absolute value than all zeros of $A(x)$. The case in which this condition is fulfilled is an important one, as we shall see in Theorem 1 and its proof, but by no means do we wish to limit our consideration to that case. Moreover, we shall find it expedient to take full advantage of the very special structure of Trench matrices by working directly with them rather than with their inverses.

We do, however, confine our attention to Hermitian Trench matrices. While the case of greatest practical interest is that of a real symmetric matrix, our results have been extended to Hermitian complex Trench matrices, as this was easily accomplished. It is hoped that someone will pursue a similar investigation for the more difficult case of non-Hermitian Trench matrices. In this connection, some fragmentary results are available. Consider, for example, a real tridiagonal Trench matrix H; this implies that both $A(x)$ and $B(x)$ are linear. If the single zero of $A(x)$ and that of $B(x)$ have the same sign, it is easily shown that H is similar to a real symmetric (tridiagonal) matrix. Thus the eigenvalues are real, and the results of this paper apply to the transformed matrix.

2. The Main Results.

It is easily seen that the Trench matrix H given by (1.1) is Hermitian if and only if $r = s$ and

$$a_\nu b_\mu = \overline{a}_\mu \overline{b}_\nu \qquad (0 \le \mu,\nu \le r) \quad . \tag{2.1}$$

If we define

$$A^*(x) = \sum_{\nu=0}^{r} \overline{a}_\nu x^\nu = \overline{A(\overline{x})} \quad , \tag{2.2}$$

then (2.1) is easily seen to be equivalent to the condition

$$B(x) = cA^*(x) \quad ,$$

where c is a nonzero real constant. If c is negative, we can consider the matrix $-H$, whose eigenvalues are, of course, the negatives of those of

H. If c is positive, $A(x)$ and $B(x)$ can be normalized so that $B(x) = A^*(x)$. Thus there is no loss of generality if we limit consideration to Hermitian Trench matrices with $B(x) = A^*(x)$.

The function

$$h(x) = A(x)\, B(1/x) = A(x)\, A^*(1/x) = \sum_{\nu=-r}^{r} h_\nu x^\nu \tag{2.3}$$

will play an important role in this paper, as it did in [4]. Consider the values of this function on the unit circle. If $x = e^{i\theta}$, it follows from (2.2) and from the fact that for this x, $x^{-1} = \bar{x}$, that

$$h(x) = A(x)\,\overline{A(x)} = |A(x)|^2 \qquad (x = e^{i\theta}) . \tag{2.4}$$

Therefore $h(x)$ is real and nonnegative on the unit circle, and moreover $h(x) = \phi(\theta)$ is a continuous periodic function of the real variable θ with period 2π. Hence it has a maximum and a minimum value, which we denote by M and m, respectively.

The following two theorems are the main results of this paper. Theorem 1 deals with the "regular" case in which (1.2) converges and Szegö's theorem applies; Theorem 2 asserts a weaker conclusion in a more general context.

<u>Theorem 1</u>. Let H_N be the Hermitian Trench matrix of order $N + 1 > 2r + 1$ characterized by the polynomials $A(x)$ and $B(x) = A^*(x)$ of degree $r > 0$. Then H_N is positive definite if and only if all the zeros of $A(x)$ are outside the unit circle. It is positive semidefinite if and only if all the zeros of $A(x)$ that are not also zeros of $A^*(1/x)$ are outside the unit circle. If it is positive definite, all its eigenvalues are greater than m and less than M, and if ρ_1 and ρ_{N+1} denote the smallest and largest eigenvalue, respectively,

$$\lim_{N\to\infty} \rho_1 = m , \qquad \lim_{N\to\infty} \rho_{N+1} = M .$$

<u>Theorem 2</u>. Let H_N denote the Hermitian Trench matrix of order $N + 1$ described in Theorem 1, and let σ_N denote its spectral radius. Then $\sigma_N < M$ for all N, and

$$\lim_{N\to\infty} \sigma_N = M .$$

3. <u>Some Implications of the Theorems</u>.

Before proceeding to the proofs of the theorems, we shall briefly

discuss a few of their implications. In some applications (see, e.g., [3]) we are interested in matrices of the form

$$G = I - kH \quad , \tag{3.1}$$

where H is a Hermitian Trench matrix and k is a positive constant. In particular we would like to know if the limit

$$G^{\infty} = \lim_{n\to\infty} G^n \tag{3.2}$$

exists. We note that Oldenburger [6] and Dresden [1] have shown that, for any square matrix G, G^{∞} exists if and only if either all the eigenvalues of G are inside the unit circle, or else $+1$ is a simple zero of the minimum polynomial of G and all other zeros are inside the unit circle. The following corollary (first conjectured by Trench) is a consequence of Theorems 1 and 2.

<u>Corollary 1</u>. Let G be given by (3.1), where H is the Hermitian Trench matrix described in Theorem 1. Then the limit (3.2) exists for all N if and only if

$$k \le 2/M \tag{3.3}$$

and no zero of $A(x)$ is inside the unit circle unless it is also a zero of $A^*(1/x)$.

<u>Proof</u>. Let (3.3) and the condition on the zeros of $A(x)$ be satisfied. Then, H is positive semidefinite by Theorem 1, since any zero of $A(x)$ <u>on</u> the unit circle is a zero of $A^*(1/x)$, and therefore its eigenvalues are nonnegative. By Theorem 2 the eigenvalues of H are less than M. Since the eigenvalues of G are obtained by subtracting from unity k times those of H, the former are greater than $1 - kM$ and not greater than 1. In fact, if H is singular, 1 is an eigenvalue of G. Since H (and therefore G) is Hermitian, all zeros of the minimum polynomial are simple, and 1 is at most a simple zero. Since $k \le 2/M$, $1 - kM \ge -1$ and so the eigenvalues of G are greater than -1. Thus, the condition of Oldenburger and Dresden is satisfied and G^{∞} exists.

On the other hand, if a zero of $A(x)$ that is not a zero of $A^*(1/x)$ is inside the unit circle, by Theorem 1, H has a negative eigenvalue. Since k is positive, this implies that G has an eigenvalue greater than 1, and so G^{∞} does not exist. Alternatively, if $A(x)$ has no zero inside the unit circle, but $k > 2/M$, then, for sufficiently large N, G has a

negative eigenvalue arbitrarily close to $1 - kM < -1$. Thus G^{∞} fails to exist for some N.

4. Proofs of the Theorems.

In these proofs we shall employ a certain special matrix notation. Let

$$P(x) = \sum_{\nu=0}^{d} p_{\nu} x^{\nu}$$

be a given polynomial. Then we define the matrix

$$P_{m,n} = (p_{ij})_{i=1}^{m}\Big|_{j=1}^{n} ,$$

where

$$p_{ij} = p_{j-i} ,$$

and it is understood that $p_{\nu} = 0$ for $\nu < 0$ and for $\nu > d$.

We shall also need to use the special matrix J_N, which is defined as the square matrix of order N having 1's on its secondary diagonal and 0's elsewhere. Note that multiplying an m by n matrix on the left by J_m reverses the order of the rows, and multiplying it on the right by J_n reverses the order of the columns. Of course, $J_N^2 = I_N$. For convenience we shall often omit the subscript of J when the context makes this clear. A persymmetric matrix Q is characterized by the fact that

$$JQJ = Q^T .$$

In the proof of Theorem 1 we shall find the case of a singular Trench matrix to be more difficult than the nonsingular case, and we shall need a lemma that expresses a singular Trench matrix as the product of a nonsingular Trench matrix and two rectangular matrices. Because singular Trench matrices appear to be interesting in their own right, the lemma is stated with more generality (i.e., without the restriction to Hermitian matrices) than is required for the purposes of this paper.

Lemma 1. Suppose the polynomial

$$E(x) = \sum_{\nu=0}^{q} e_{\nu} x^{\nu}$$

divides both $A(x)$ and $x^s B(1/x)$ and define

$$E^{\#}(x) = x^q E(1/x) ,$$

(4.1)
$$\hat{A}(x) = A(x)/E(x) = \sum_{\nu=0}^{r-q} \hat{a}_\nu x^\nu$$

and

(4.2)
$$\hat{B}(x) = B(x)/E^{\#}(x) = \sum_{\mu=0}^{s-q} \hat{b}_\mu x^\mu \quad .$$

Let H be the Trench matrix of order $N + 1$ characterized by $A(x)$ and $B(x)$ as in (1.1), and let D be the Trench matrix of order $N - q + 1$ characterized by $\hat{A}(x)$ and $\hat{B}(x)$; thus, the generating function of the elements of the ith row of D is

(4.3)
$$D_i(x) = \begin{cases} x^i \hat{A}(x) \sum_{\mu=0}^{i} \hat{b}_\mu x^{-\mu} & (0 \le i < s - q) \\ x^i \hat{A}(x) \hat{B}(1/x) & (s - q \le i \le N - r) \\ x^i \hat{B}(1/x) \sum_{\nu=0}^{N-i} \hat{a}_\nu x^\nu & (N - r < i \le N - q). \end{cases}$$

Then,

(4.4)
$$H = E^{\#T}_{N-q+1,N+1} D E_{N-q+1,N+1} \quad .$$

Proof. For convenience let us drop the subscripts of the rectangular matrices in (4.4). It follows from (4.3) and from the structure of E that the generating function of the elements of the ith row of DE is $D_i(x) E(x)$. (Note that D is of order $N - q + 1$, and that the "special rows" at the bottom of D are $r - q$ in number, and $(N - q + 1) - (r - q) = N - r + 1$.)

With the understanding that $b_\mu = 0$ for $\mu > s$, the first two parts of (1.1) can both be written in the form

(4.5)
$$H_i(x) = A(x) \sum_{\mu=0}^{i} b_\mu x^{i-\mu} = A(x) \sum_{\mu=0}^{i} b_{i-\mu} x^\mu \quad .$$

Thus, by (4.1) and (4.2) we have

(4.6)
$$D_i(x) = \hat{A}(x) \sum_{\mu=0}^{i} \hat{b}_{i-\mu} x^\mu \quad .$$

Therefore, the generating function of the elements of the ith row of $E^{\#T} DE$ is

$$\sum_{k=0}^{i} e_{q-i+k} D_k(x) E(x) = A(x) \sum_{k=0}^{i} e_{q-i+k} \sum_{\mu=0}^{k} \hat{b}_{k-\mu} x^{\mu}$$

by (4.1) and (4.6). Reversing the order of summation gives

$$A(x) \sum_{\mu=0}^{i} x^{\mu} \sum_{k=\mu}^{i} e_{q-i+k} \hat{b}_{k-\mu}, \tag{4.7}$$

and the summation with respect to k can be rewritten as

$$\sum_{\nu=0}^{i-\mu} e_{q-\nu} \hat{b}_{i-\mu-\nu} = b_{i-\mu}$$

by (4.2). Thus (4.7) reduces finally to

$$A(x) \sum_{\mu=0}^{i} b_{i-\mu} x^{\mu} = H_i(x)$$

by (4.5). This proves (4.4) for rows 0 to $N - r$, inclusive, of H.

Let us now consider the matrix JHJ, in which the order of both rows and columns of H is reversed. By means of (1.1) it is not difficult to see that this is a Trench matrix in which, as compared with H, the roles of $A(x)$ and $B(x)$ are interchanged. Therefore by the first part of this proof, the equation

$$JHJ = E^T (JDJ) E^{\#} \tag{4.8}$$

holds for rows 0 to $N - s$, inclusive, of the matrices on both sides. Now, it is easily verified that $JE^T J = E^{\#T}$ and $JE^{\#} J = E$. Thus, multiplying (4.8) by J both on the left and on the right gives (4.4). As rows 0 to $N - s$ of JHJ become rows s to N of H (with the order of the elements reversed), this completes the proof of the lemma.

Proof of Theorem 1. This proof consists of three parts. First, we shall use Szegö's theorem to show that if all the zeros of $A(x)$ are outside the unit circle, then H is positive definite, and the inequalities and limiting relations for the eigenvalues follow. Second, we shall prove that if $A(x)$ has one or more zeros on or inside the unit circle that are also zeros of $x^r A^*(1/x)$ (but all other zeros are outside the unit circle), then H is positive semidefinite. Finally, we shall show that if $A(x)$ has a zero inside the unit circle that is not a zero of $x^r A^*(1/x)$ as well, H is not positive definite or semidefinite.

Let all the zeros of $A(x)$ be outside the unit circle. Then the zeros of $x^s B(1/x) = x^r A^*(1/x)$ are all inside the unit circle, and it was shown in [4] that $[h(x)]^{-1} = [A(x)\ A^*(1/x)]^{-1}$ has a Laurent expansion (1.2) that converges in an annular region containing the unit circle. It follows from the discussion preceding Theorem 1 that $[h(x)]^{-1}$ is real and positive on the unit circle, its maximum and minimum values there being $1/m$ and $1/M$, respectively. Therefore, by Szegö's theorem (see Chapter 5 of [2]) the eigenvalues of $T_N = H_N^{-1}$ are greater than $1/M$ and less than $1/m$ for all N, and these bounds are the limits of the smallest and the largest eigenvalues as N goes to infinity. As the eigenvalues of H_N are the reciprocals of those of T_N, the statements in Theorem 1 concerning the positive definite case follow at once.

In order to deal with the case in which H is singular, we specialize the formula (4.4) established in Lemma 1. We recall that the zeros of $B(1/x) = A^*(1/x)$ are the conjugates of the reciprocals of those of $A(x)$. Let all the zeros of $A(x)$ that are not also zeros of $A^*(1/x)$ be outside the unit circle. In fact, since the conjugate of a point on the unit circle is also its reciprocal, any zero of $A(x)$ that is on the unit circle is also a zero of $A^*(1/x)$. Therefore, let $A(x) = \hat{A}(x)\ E(x)$, where the zeros of $\hat{A}(x)$ are those of $A(x)$ that are outside the unit circle, and the zeros of $E(x)$ are those of $A(x)$ that are also zeros of $A^*(1/x)$. It follows that $E^*(x)$ and $E^{\#}(x)$ have the same zeros, and are therefore identical. Since $E^{\#}(x)$ is obtained from $E(x)$ by reversing the order of the coefficients, and $E^*(x)$ by taking the conjugates of the coefficients, we must have

$$e_{q-\nu} = \bar{e}_\nu \qquad (\nu = 0, 1, \ldots, q) .$$

It follows that $E^{\#T} = E^{CT}$, and (4.4) becomes

$$H_N = E^{CT}_{N-q+1,N+1}\ DE_{N-q+1,N+1} \tag{4.9}$$

If u is an arbitrary nonzero vector, and $v = Eu$, then by (4.9), $u^{CT} H_N u = v^{CT} Dv$, which is nonnegative, since D is positive definite. Therefore, H is positive semidefinite.

We come finally to the third part of the proof. Let $A(x)$ have a zero, $x = \xi$, inside the unit circle such that $A^*(\xi^{-1}) \neq 0$. Since H is a Trench matrix, $a_0 \neq 0$, and so $\xi \neq 0$. It follows that $\bar{\xi}^{-1}$ (which is of course outside the unit circle) is a zero of $A^*(1/x)$. Now let v be the vector whose ith component (starting the numbering with 0) is $\bar{\xi}^{-i}$. It follows from the definition of the generating function that the ith component of Hv is $H_i(\bar{\xi}^{-1})$. For all but the first r components (i.e., those numbered from 0 to $r-1$), $A^*(\bar{\xi}) = 0$ is a factor of $H_i(\bar{\xi}^{-1})$, and so these components vanish. For $0 \leq i < r$,

$$H_i(\bar{\xi}^{-1}) = A(\bar{\xi}^{-1}) \sum_{\mu=0}^{i} \bar{a}_\mu \bar{\xi}^{\mu-i} \quad . \tag{4.10}$$

Now, let the polynomial

$$F(x) = \sum_{\nu=0}^{r-1} f_\nu x^\nu$$

be defined by

$$A(x) = (x - \xi)F(x) = -\xi(1 - x\xi^{-1})F(x) \quad . \tag{4.11}$$

Then,

$$F(x) = -\xi^{-1}(1 - x\xi^{-1})^{-1} A(x) \quad ,$$

and consequently,

$$f_j = -\sum_{\nu=0}^{j} a_\nu \xi^{\nu-j-1} = -\xi^{-j-1} \sum_{\nu=0}^{j} a_\nu \xi^\nu \qquad (0 \leq j < r) \quad ,$$

or

$$\sum_{\nu=0}^{j} a_\nu \xi^{\nu-j} = -\xi f_j \quad . \tag{4.12}$$

Substitution of (4.11) and the conjugate of (4.12) in (4.10) gives

$$H_i(\bar{\xi}^{-1}) = -\bar{\xi}(\bar{\xi}^{-1} - \xi)F(\bar{\xi}^{-1})\bar{f}_i = (\xi\bar{\xi} - 1)F(\bar{\xi}^{-1})\bar{f}_i \quad ,$$

and so

$$v^{CT} Hv = \sum_{i=0}^{r-1} \xi^{-i} H_i(\bar{\xi}^{-1}) = (\xi\bar{\xi} - 1)F(\bar{\xi}^{-1}) \overline{F(\bar{\xi}^{-1})} \quad ,$$

an expression which is clearly negative, since $|\xi| < 1$ and

$$0 \neq \overline{A^*(\bar{\xi}^{-1})} = A(\bar{\xi}^{-1}) = (\bar{\xi}^{-1} - \xi)F(\bar{\xi}^{-1}) ,$$

so that $F(\bar{\xi}^{-1}) \neq 0$. It follows that H is not positive definite or semidefinite. This completes the proof of Theorem 1.

In the proof of Theorem 2 we shall need a lemma that expresses the Hermitian Trench matrix H in terms of simpler matrices. Let us define $\tilde{A}(x)$ by

$$\tilde{A}(x) = x^r A^*(1/x) ,$$

and let us define $\tilde{A}$ as the square matrix of order $N + 1$

$$\tilde{A} = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{A}_{r,r} \end{bmatrix} . \tag{4.13}$$

Then we have

Lemma 2. If H is the Hermitian Trench matrix defined in Theorem 1,

$$H = A^{CT}_{N+1,N+1} A_{N+1,N+1} - \tilde{A}^{CT} \tilde{A} . \tag{4.14}$$

Proof. First we note that H and the first term of the right member of (4.14) agree in all their elements except the square submatrices of order r in the lower right corner. For all but the last r rows, this follows easily from (1.1) taking $B(x) = A^*(x)$. For the last r rows, excluding the square submatrix in the right corner, it follows from the Hermitian symmetry of both matrices. Moreover, the second term of the right member of (4.14) has zeros everywhere except in the corner submatrix mentioned. These observations make (4.14) at least plausible, and permit us to limit our attention to the r by r submatrices in the lower right corner.

In the case of the first term of the right member of (4.14), this corner submatrix is obtained by multiplying the last r rows of the first factor by the last r columns of the second factor. Taking into account that some of the elements of these rows and columns are zeros, this product can be written as

$$\tilde{A}_{r,2r} \tilde{A}^{CT}_{r,2r} .$$

Moreover, by partitioning the first factor of this latter product into the first r columns and the last r columns, and the second factor similarly by rows, we obtain

$$\tilde{A}_{r,2r} \tilde{A}^{CT}_{r,2r} = \tilde{A}_{r,r} \tilde{A}^{CT}_{r,r} + A^{CT}_{r,r} A_{r,r} ,$$

or

$$A^{CT}_{r,r}\, A_{r,r} = \tilde{A}_{r,2r}\, \tilde{A}^{CT}_{r,2r} - \tilde{A}_{r,r}\, \tilde{A}^{CT}_{r,r} \quad . \tag{4.15}$$

Now, the left member of (4.15) is precisely the square submatrix of order r in the upper left corner of H. Since H is Hermitian and persymmetric, the one in the lower right corner is obtained from it by reversing the order of both rows and columns and then taking the conjugate. Accordingly, let us perform these operations on the right member. As the first term is Hermitian and Toeplitz, the effect of the operations is to leave that term unchanged. Coming now to the second term, since

$$J_r\, \tilde{A}_{r,r}\, \tilde{A}^{CT}_{r,r}\, J_r = J_r\, \tilde{A}_{r,r}\, J_r\, J_r\, \tilde{A}^{CT}_{r,r}\, J_r \quad ,$$

we can perform the operations on each factor separately. We note also that the effect of the three operations on a matrix $P_{r,r}$ is to take the conjugate transpose. Thus the result is $\tilde{A}^{CT}_{r,r}\, \tilde{A}_{r,r}$. In view of (4.13), this proves (4.14).

Proof of Theorem 2. Let us denote by K and L the respective products in the right member of (4.14), so that

$$H = K - L \quad .$$

Clearly K is Hermitian positive definite and L is Hermitian positive semidefinite. Let v be an arbitrary nonzero vector of complex elements. Then the Rayleigh quotients satisfy

$$\frac{v^{CT} Hv}{v^{CT} v} = \frac{v^{CT} Kv}{v^{CT} v} - \frac{v^{CT} Lv}{v^{CT} v} \leq \frac{v^{CT} Kv}{v^{CT} v} \quad . \tag{4.16}$$

Let

$$V(t) = \sum_{\nu=0}^{N} v_\nu\, e^{i\nu t}$$

be the characteristic function of v. Then,

$$A(e^{-it})\, V(t) = \sum_{\nu=-r}^{N} w_\nu\, e^{i\nu t} \quad ,$$

where, for $0 \leq \nu \leq N$, w_ν is the νth component of $A_{N+1,N+1}\, v$. (It may be helpful to the reader to think of the vector v as being extended by annexing a number of zeros at the bottom.) By Parseval's formula (see [7], p. 699)

$$v^{CT} v = |v|^2 = \frac{1}{2\pi}\int_0^{2\pi} |V(t)|^2 \, dt \quad ,$$

while

$$v^{CT} Kv = |A_{N+1,N+1} v|^2 \leq \frac{1}{2\pi}\int_0^{2\pi} |A(e^{-it}) \, V(t)|^2 \, dt \quad .$$

But by (2.4), $|A(e^{-it})|^2 = h(e^{-it})$, and therefore

(4.17) $$v^{CT} Kv \leq \frac{1}{2\pi}\int_0^{2\pi} h(e^{-it}) |V(t)|^2 \, dt < M \; v^{CT} v \quad .$$

Note that since the zeros of $V(t)$ are a set of measure zero, the second inequality of (4.17) could be replaced by equality only if $h(x)$ is a constant function, which would imply that H is diagonal, and therefore a scalar matrix. This is tantamount to saying that $r = 0$, contrary to hypothesis (see description of H in Theorem 1).

It follows from (4.16) and (4.17) that

(4.18) $$\frac{v^{CT} Hv}{v^{CT} v} < M \quad ,$$

and, since the greatest eigenvalue of H is the maximum value of the Rayleigh quotient in the left member of (4.18), we have shown that the greatest eigenvalue of H is less than M.

However, it will be noted that in Theorem 2 we have not imposed the condition that would make H positive definite or semidefinite. Thus H may have negative eigenvalues, and it is conceivable that such a negative eigenvalue might exceed M in absolute value. We must prove that this is not the case. The algebraically smallest eigenvalue is the minimum value of the Rayleigh quotient in the left member of (4.18). Since K and L are both Hermitian positive semidefinite, this minimum value is greater than, or at least equal to the negative of the maximum value of the Rayleigh quotient with respect to L.

It follows from (4.13) and (4.14) that the elements of L are all zero with the exception of the square submatrix of order r in the lower right corner. Because of this structure, the eigenvalues of L, other than zero, are those of $\hat{L} = \tilde{A}^{CT}_{r,r} \tilde{A}_{r,r}$. Therefore the maximum Rayleigh quotient with respect to $\hat{L}$ is the same as that with respect to L. Therefore, a lower

bound to the eigenvalues of H is $-\hat{\rho}$, where $\hat{\rho}$ is the largest eigenvalue of $\hat{L}$. In order to complete the proof that the spectral radius of H is less than M, we must show that $\hat{\rho} < M$. In fact, if v is an arbitrary vector of r complex components and its characteristic function is

$$V(t) = \sum_{\nu=0}^{r-1} v_\nu e^{i\nu t} ,$$

then, by reasoning closely parallel to that used in the first part of this proof, we conclude that

$$v^{CT} \hat{L} v \leq \frac{1}{2\pi} \int_0^{2\pi} |A^*(e^{-it}) V(t)|^2 \, dt < M v^{CT} v .$$

Since $\hat{L}$ is Hermitian, $\hat{\rho}$ is the maximum value of the Rayleigh quotient.

To prove the second part we let M' be an arbitrary positive constant less than M and show that, for a suitable vector v and for sufficiently large N, the Rayleigh quotient $v^{CT} H_N v / v^{CT} v$ can be made larger than M'. Since $h(e^{it})$ is a continuous function of t and M is its maximum value in $[0, 2\pi]$, there is some value $t = \tau$, such that

$$h(e^{i\tau}) > M' .$$

Let us choose $v = [v_0, v_1, \ldots, v_N]^T$ so that $v_\nu = e^{-i\nu\tau}$ for $0 \leq \nu \leq N$. Then, except for the first r and the last r components, the νth component of Hv is $h(e^{i\tau}) v_\nu$. Therefore

$$v^{CT} Hv = (N - 2r + 1) h(e^{i\tau}) + C , \tag{4.19}$$

where C is the contribution of the first r and the last r components. Since every component of v has absolute value 1, an upper bound to the absolute value of C is the sum of the absolute values of the elements in the first r and the last r rows of H. Call this C', and note that C' does not depend on N.

Now choose N sufficiently large so that

$$N + 1 > \frac{C' + 2r\, h(e^{i\tau})}{h(e^{i\tau}) - M'} .$$

Then

$$(N + 1)[h(e^{i\tau}) - M'] > C' + 2r\, h(e^{i\tau}) ,$$

or

$$(N - 2r + 1)h(e^{i\tau}) > (N + 1)M' + |C| \quad ,$$

and consequently

$$(N - 2r + 1)h(e^{i\tau}) + C > (N + 1)M' \quad . \tag{4.20}$$

Since $|v_\nu|^2 = 1$ for every ν,

$$v^{CT} v = N + 1$$

and therefore by (4.19) and (4.20)

$$\frac{v^{CT} Hv}{v^{CT} v} > M'$$

as required. This completes the proof of Theorem 2.

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A banded matrix $H = (h_{ij})_{i,j=0}^{N}$ is one such that $h_{ij} = 0$ for $j - i > r$ and for $i - j > s$, where r and s are nonnegative integers. In [5] W. F. Trench and I called it strictly banded if, in addition, $r + s \le N$. We also showed that a necessary condition for a strictly banded matrix to have a Toeplitz inverse is that it have a certain special structure fully characterized by two polynomials, $A(x)$ of degree r and $B(x)$ of degree s. I

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call a matrix having this special structure a Trench matrix. It was also shown in [5] that a Trench matrix is nonsingular if and only if $A(x)$ and $B(x)$ have no common zero, and that a strictly banded matrix has a Toeplitz inverse if and only if it is a nonsingular Trench matrix. In this paper there are established bounds for eigenvalues of Hermitian Trench matrices that depend only on the polynomials $A(x)$ and $B(x)$ and not on the order of the matrix.